Three Cube Roots of Unity & Four Fourth Roots of Unity
Three Cube Roots & Four Fourth Roots of a Number
Exercise 4.4

1. Find the three cube roots of: 8, -8, 27, -27, and 64

(I) 8
Solution:
Let x be a cube root of 8

$\therefore\ x = \sqrt[3]{8} = (8)^{\frac{1}{3}}$

$x^3 = ((8)^{\frac{1}{3}})^3$

$x^3 = 8$

$x^3 - 8 = 0$

$(x)^3 - (2)^3 = 0$

$(x-2)(x^2+2x+4) = 0$

$x - 2 = 0$ $\qquad$ $x^2 + 2x + 4 = 0$

$x = 2$ $\qquad$ $x = \dfrac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$

$x = \dfrac{-2 \pm \sqrt{-12}}{2}$

$x = \dfrac{-2 \pm \sqrt{-(4)(3)}}{2}$

$x = \dfrac{-2 \pm 2\sqrt{-3}}{2}$

$x = \dfrac{2(-1 \pm \sqrt{3}i)}{2}$ $\qquad \Rightarrow \qquad$ $x = 2(\dfrac{-1 \pm \sqrt{3}i}{2})$

$x = 2(\dfrac{-1 + \sqrt{3}i}{2})$ $\qquad$ $x = 2(\dfrac{-1 - \sqrt{3}i}{2})$

$x = 2w$ $\qquad$ $x = 2w^2$ $\qquad$ Q $w = \dfrac{-1 + \sqrt{3}i}{2}$

Q $w^2 = \dfrac{-1 - \sqrt{3}i}{2}$

Hence, three cube roots of 8 are: $\{2, 2w, 2w^2\}$

(II) -8
Solution:
Let x be a cube root of -8

$\therefore\ x=\sqrt[3]{-8}=(-8)^{\frac{1}{3}}$

$x^3=((-8)^{\frac{1}{3}})^3$

$x^3=-8$

$x^3+8=0$

$(x)^3+(2)^3=0$

$(x+2)(x^2-2x+4)=0$

$x+2=0$ $\qquad$ $x^2-2x+4=0$

$x=-2$ $\qquad$ $x=\dfrac{-(-2)\pm\sqrt{(-2)^2-4(1)(4)}}{2(1)}$

$x=\dfrac{2\pm\sqrt{-12}}{2}$

$x=\dfrac{2\pm\sqrt{-(4)(3)}}{2}$

$x=\dfrac{2\pm2\sqrt{-3}}{2}$

$x=\dfrac{2(1\pm\sqrt{3}i)}{2}$ $\qquad\Rightarrow\qquad$ $x=2(\dfrac{1\pm\sqrt{3}i}{2})$

$x=2(\dfrac{1+\sqrt{3}i}{2})$ $\qquad$ $x=2(\dfrac{1-\sqrt{3}i}{2})$

$x=-2(\dfrac{-1-\sqrt{3}i}{2})$ $\qquad$ $x=-2(\dfrac{-1+\sqrt{3}i}{2})$

$x=-2w^2$ $\qquad$ $x=-2w$ $\qquad$ Q $w=\dfrac{-1+\sqrt{3}i}{2}$

Q $w^2=\dfrac{-1-\sqrt{3}i}{2}$

Hence, three cube roots of -8 are: $\{-2,-2w,-2w^2\}$

(III) 27
Solution:
Let x be a cube root of 27

$\therefore\ x=\sqrt[3]{27}=(27)^{\frac{1}{3}}$

$x^3=((27)^{\frac{1}{3}})^3$

$x^3=27$

$x^3-27=0$

$(x)^3-(3)^3=0$

$(x-3)(x^2+3x+9)=0$

$x-3=0$ $\qquad$ $x^2+3x+9=0$

$x=3$ $\qquad$ $x=\dfrac{-3\pm\sqrt{(3)^2-4(1)(9)}}{2(1)}$

$x=\dfrac{-3\pm\sqrt{-27}}{2}$

$x=\dfrac{-3\pm\sqrt{-(9)(3)}}{2}$

$x=\dfrac{-3\pm3\sqrt{-3}}{2}$

$x=\dfrac{3(-1\pm\sqrt{-3})}{2}$

$x=\dfrac{3(-1\pm\sqrt{3}i)}{2}$ $\qquad\Rightarrow\qquad$ $x=3(\dfrac{-1\pm\sqrt{3}i}{2})$

$x=3(\dfrac{-1+\sqrt{3}i}{2})$ $\qquad$ $x=3(\dfrac{-1-\sqrt{3}i}{2})$

$x=3w$ $\qquad$ $x=3w^2$

Hence, three cube roots of 27 are: $\{3,3w,3w^2\}$

(IV) -27

Solution:

Let x be a cube root of -27

$\therefore\ x=\sqrt[3]{-27}=(-27)^{\frac{1}{3}}$

$x^3=((-27)^{\frac{1}{3}})^3$

$x^3=-27$

$x^3+27=0$

$(x)^3+(3)^3=0$

$(x+3)(x^2-3x+9)=0$

$x+3=0$ $\qquad$ $x^2-3x+9=0$

$x=-3$ $\qquad$ $x=\dfrac{-(-3)\pm\sqrt{(-3)^2-4(1)(9)}}{2(1)}$

$x=\dfrac{3\pm\sqrt{-27}}{2}$

$x=\dfrac{3\pm\sqrt{-(9)(3)}}{2}$

$$x=\frac{3\pm 3\sqrt{-3}}{2}$$
$$x=\frac{3(1\pm\sqrt{-3})}{2}$$
$$x=\frac{3(1\pm\sqrt{3}i)}{2} \qquad \Rightarrow \qquad x=3(\frac{1\pm\sqrt{3}i}{2})$$
$$x=3(\frac{1+\sqrt{3}i}{2}) \qquad x=3(\frac{1-\sqrt{3}i}{2})$$
$$x=-3(\frac{-1-\sqrt{3}i}{2}) \qquad x=-3(\frac{-1+\sqrt{3}i}{2})$$
$$x=-3w^2 \qquad x=-3w$$

Hence, three cube roots of -27 are: $\{-3,-3w,-3w^2\}$

(V) 64

Solution:

Let x be a cube root of 64

$\therefore\ x=\sqrt[3]{64}=(64)^{\frac{1}{3}}$

$x^3=((64)^{\frac{1}{3}})^3$

$x^3=64$

$x^3-64=0$

$(x)^3-(4)^3=0$

$(x-4)(x^2+4x+16)=0$

$x-4=0 \qquad x^2+4x+16=0$

$x=4 \qquad x=\frac{-4\pm\sqrt{(4)^2-4(1)(16)}}{2(1)}$

$$x=\frac{-4\pm\sqrt{-48}}{2}$$
$$x=\frac{-4\pm\sqrt{-(16)(3)}}{2}$$
$$x=\frac{-4\pm 4\sqrt{-3}}{2}$$
$$x=\frac{4(-1\pm\sqrt{3}i)}{2} \qquad \Rightarrow \qquad x=4(\frac{-1\pm\sqrt{3}i}{2})$$
$$x=4(\frac{-1+\sqrt{3}i}{2}) \qquad x=4(\frac{-1-\sqrt{3}i}{2})$$
$$x=4w \qquad x=4w^2$$

Hence, three cube roots of 64 are: $\{4,4w,4w^2\}$

2. Evaluate:

(i) $(1+w-w^2)^8$ (ii) $w^{28}+w^{29}+1$ (iii) $(1+w-w^2)(1-w+w^2)$

(iv) $(\frac{-1+\sqrt{-3}}{2})^9+(\frac{-1-\sqrt{-3}}{2})^7$ (v) $(-1+\sqrt{-3})^5+(-1-\sqrt{3})^5$

(i) $(1+w-w^2)^8$

Solution:

$(1+w-w^2)^8$

Add and subtract w^2

$$\begin{aligned}(1+w-w^2)^8 &= (1+w+w^2-w^2-w^2)^8 \\ &= (1+w+w^2-2w^2)^8 \\ &= (0-2w^2)^8 && \because 1+w+w^2=0 \\ &= 256w^{16} \\ &= 256(w^3)^{15}w \\ &= 256(1)^{15}w && \because w^3=1 \\ &= 256w\end{aligned}$$

(ii) $w^{28}+w^{29}+1$

Solution:

$$\begin{aligned}w^{28}+w^{29}+1 &= (w^3)^9w+(w^3)^9w^2+1 \\ &= (1)^9w+(1)^9w^2+1 \\ &= w+w^2+1 \\ &= 0 && \because w+w^2+1=1+w+w^2=0\end{aligned}$$

(iii) $(1+w-w^2)(1-w+w^2)$

Solution:

$(1+w-w^2)(1-w+w^2)$

Add and subtract w^2 in the 1st bracket, and add and subtract w in the 2nd bracket.

$$\begin{aligned}(1+w-w^2)(1-w+w^2) &= (1+w+w^2-w^2-w^2)(1+w-w-w+w^2) \\ &= (1+w+w^2-2w^2)(1+w+w^2-2w) \\ &= (0-2w^2)(0-2w) \\ &= (-2w^2)(-2w) \\ &= 4w^3 \\ &= 4\end{aligned}$$

(iv) $(\frac{-1+\sqrt{-3}}{2})^9+(\frac{-1-\sqrt{-3}}{2})^7$

Solution:

$$(\frac{-1+\sqrt{-3}}{2})^9+(\frac{-1-\sqrt{-3}}{2})^7 = (w)^9+(w^2)^7 \qquad \text{Q } w=\frac{-1+\sqrt{-3}}{2},\ w^2=\frac{-1-\sqrt{-3}}{2}$$

$$= (w^3)^3+w^{14}$$
$$= (w^3)^3+(w^3)^4 w^2$$
$$= (1)^3+(1)^4 w^2$$
$$= 1+w^2$$
$$= -w$$

If $(\frac{-1+\sqrt{-3}}{2})^7+(\frac{-1-\sqrt{-3}}{2})^7$

Then solution:

$$(\frac{-1+\sqrt{-3}}{2})^7+(\frac{-1-\sqrt{-3}}{2})^7 = (w)^7+(w^2)^7$$
$$= (w^3)^2 w+w^{14}$$
$$= (w^3)^2 w+(w^3)^4 w^2$$
$$= w+w^2$$
$$= -1$$

(v) $(-1+\sqrt{-3})^5+(-1-\sqrt{3})^5$

Solution:

Multiply and divide by "2" with both the brackets.

$$(-1+\sqrt{-3})^5+(-1-\sqrt{3})^5 = (2\times\frac{-1+\sqrt{-3}}{2})^5+(2\times\frac{-1-\sqrt{-3}}{2})^5$$
$$= (2\times w)^5+(2\times w^2)^5$$
$$= 32w^5+32w^{10}$$
$$= 32w^3w^2+32(w^3)^3 w$$
$$= 32(1)w^2+32(1)w$$
$$= 32w^2+32w$$
$$= 32(w^2+w)$$
$$= 32(-1)$$
$$= -32$$

3. Show that:

(i) $x^3-y^3=(x-y)(x-wy)(x-w^2y)$

Solution:

$$x^3-y^3=(x-y)(x-wy)(x-w^2y)$$
$$x^3-y^3=(x^2-wxy-xy+wy^2)(x-w^2y)$$
$$x^3-y^3=(x^3-w^2x^2y-wx^2y+w^3xy^2-x^2y+w^2xy^2+wxy^2-w^3y^3)$$

Rearranging we have,

$x^3 - y^3 = (x^3 - w^2x^2y - wx^2y - x^2y + w^3xy^2 + w^2xy^2 + wxy^2 - w^3y^3)$
$x^3 - y^3 = x^3 - x^2y(w^2 + w + 1) + xy^2(w^3 + w^2 + w) - w^3y^3$
$x^3 - y^3 = x^3 - x^2y(w^2 + w + 1) + wxy^2(w^2 + w + 1) - w^3y^3$
$x^3 - y^3 = x^3 - w^3y^3$
$x^3 - y^3 = x^3 - y^3$ Proved.

(ii) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + wy + w^2z)(x + w^2y + wz)$
Solution:
$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + wy + w^2z)(x + w^2y + wz)$
Take L.H.S
$= (x + y + z)(x + wy + w^2z)(x + w^2y + wz)$
$= (x + y + z)(x^2 + w^2xy + wxz + wxy + w^3y^2 + w^2yz + w^2xz + w^4yz + w^3z^2)$
$= (x + y + z)(x^2 + xy(w^2 + w) + xz(w + w^2) + yz(w^2 + w) + w^3y^2$
$= (x + y + z)(x^2 + xy(-1) + xz(-1) + yz(-1) + (1)y^2$
$= (x + y + z)(x^2 - xy - xz - yz + y^2)$
$= x^3 - x^2y - x^2z - xyz + xy^2 + x^2y - xy^2 - xyz - y^2z + y^3 + x^2z - xyz - xz^2 - yz^2 + y^2z$
$= x^3 + y^3 + z^3 - 3xyz$ Proved.

(iii) $(1 + w)(1 + w^2)(1 + w^4)(1 + w^8)$ …… $2n$ factors = 1
Solution:
$(1 + w)(1 + w^2)(1 + w^4)(1 + w^8)$ …… $2n$ factors
$(1 + w)(1 + w^2)(1 + w)(1 + w^2)$ …… $2n$ factors
$(1 + w^2 + w + w^3)(1 + w^2 + w + w^3)$ …… $2n$ factors
$(1 + w^2 + w + 1)(1 + w^2 + w + 1)$ …… $2n$ factors
$(0 + 1)(0 + 1)$ …… $2n$ factors
$(1)(1)$ …… $2n$ factors
$(1)^{2n}$
1 Proved.

4. If w is a root of $x^2 + x + 1 = 0$, show that its other root is w^2 and prove that $w^3 = 1$.
Solution:
If w is the root of the equation $x^2 + x + 1 = 0$ then,
$w^2 + w + 1 = 0$ $\rightarrow$ Equation (A)
Now, to prove that its other root is w^2, Replace x by w^2,
$x^2 + x + 1 = 0 \Rightarrow (w^2)^2 + w^2 + 1 = 0$
Add and subtract w^2.
$(w^2)^2 + w^2 + w^2 + 1 - w^2 = 0$
$(w^2)^2 + 2w^2 + 1 - w^2 = 0$

$(w^2+1)^2 - w^2 = 0$

$(w^2+1+w)(w^2+1-w) = 0$ $\qquad$ Q $a^2 - b^2 = (a+b)(a-b)$

From Equation (A) $w^2+1+w=0$, therefore

$(0)(w^2+1-w) = 0$

$\Rightarrow$ $(w^2)^2 + w^2 + 1 = 0$ $\rightarrow$ Equation (B)

Hence, w^2 is also a root of $x^2+x+1=0$.

Now, to prove $w^3=1$ subtract Equation (A) from B.

$$\begin{array}{l} (w^2)^2 + w^2 + 1 = 0 \\ \underline{-\ w^2 \pm w \pm 1 = 0} \\ w^4 - w = 0 \end{array}$$

$w(w^3-1) = 0$

As, $w \neq 0$ $\qquad$ $w^3 - 1 = 0$

$w^3 = 1$ $\qquad$ Proved.

5. Prove that complex cube roots of -1 are $\frac{1+\sqrt{3}i}{2}$ and $\frac{1-\sqrt{3}i}{2}$; and hence prove that

$$\left(\frac{1+\sqrt{-3}}{2}\right)^9 + \left(\frac{1-\sqrt{-3}}{2}\right)^9 = -2$$

Solution:

Let x be the cube root of -1, therefore

$x = \sqrt[3]{-1} = (-1)^{\frac{1}{3}}$

$x^3 = -1$

$x^3 + 1 = 0$

$(x)^3 + (1)^3 = 0$

$(x+1)(x^2 - x + 1) = 0$

$x+1=0$ $\qquad$ $x^2 - x + 1 = 0$

$x = -1$ $\qquad$ $x = \frac{-(-1) \pm \sqrt{1-4}}{2(1)}$

$x = \frac{1 \pm \sqrt{-3}}{2}$

$x = \frac{1 \pm \sqrt{3}i}{2}$

$x = \frac{1+\sqrt{3}i}{2}$ $\qquad$ $x = \frac{1-\sqrt{3}i}{2}$

Hence, three cube roots of -1 are: $\{-1, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}\}$

Now, we have to prove, $\left(\frac{1+\sqrt{-3}}{2}\right)^9+\left(\frac{1-\sqrt{-3}}{2}\right)^9=-2$ $\rightarrow$ Equation (C)

As we know that: $w=\frac{-1+\sqrt{-3}}{2}$ and $w^2=\frac{-1-\sqrt{-3}}{2}$

$-w=\frac{1-\sqrt{-3}}{2}$ $\quad$ $-w^2=\frac{1+\sqrt{-3}}{2}$

Hence, equation (C) becomes:

$\left(-w^2\right)^9+\left(-w\right)^9=-2$

Take R.H.S

$\left(-w^2\right)^9+\left(-w\right)^9$

$-w^{18}-w^9 = -(w^3)^6-(w^3)^3 = -(1)^6-(1)^3 = -2$ Proved.

6. If w is a cube root of unity, form an equation whose roots are $2w$ and $2w^2$.

Solution:

To form an equation, whose roots are $2w$ and $2w^2$, we have

$(x-2w)(x-2w^2)=0$ $\quad$ Q If $2w$ is a root $\Rightarrow x=2w \Rightarrow x-2w=0$

Is $2w^2$ is a root $\Rightarrow x=2w^2 \Rightarrow x-2w^2=0$

$x^2-2w^2x-2wx+4w^3=0$

$x^2-2x(w^2+w)+4w^3=0$

$x^2-2x(-1)+4(1)=0$ $\quad$ Q $w^2+w=-1,\ w^3=1$

$x^2+2x+4=0$ Answer

7. Find four fourth roots of: 16, 81, and 625

(i) 16

Solution:

Let x be a fourth root of 16, therefore,

$x=\sqrt[4]{16}$

$x=16^{\frac{1}{4}}$

$(x)^4=\left(16^{\frac{1}{4}}\right)^4$

$x^4=16$

$x^4-16=0$

$\left(x^2\right)^2-\left(4\right)^2=0$ $\quad$ Q $a^2-b^2=(a+b)(a-b)$

$(x^2+4)(x^2-4)=0$

$x^2+4=0$ $\quad$ $x^2-4=0$

$x^2=-4$ $\quad$ $x^2=4$

$x=\pm\sqrt{-4}$ $\quad$ $x=\pm\sqrt{4}$

$x = \pm\sqrt{4}i$ $\qquad$ $x = \pm 2$

$x = \pm 2i$

Hence four fourth roots of 16 are: $2, -2, 2i, -2i$

(ii) 81

Solution:

Let x be a fourth root of 81, therefore,

$x = \sqrt[4]{81}$

$x = 81^{\frac{1}{4}}$

$(x)^4 = \left(81^{\frac{1}{4}}\right)^4$

$x^4 = 81$

$x^4 - 81 = 0$

$\left(x^2\right)^2 - (9)^2 = 0$

$(x^2+9)(x^2-9) = 0$ $\qquad$ Q $a^2 - b^2 = (a+b)(a-b)$

$x^2 + 9 = 0$ $\qquad$ $x^2 - 9 = 0$

$x^2 = -9$ $\qquad$ $x^2 = 9$

$x = \sqrt{-9}$ $\qquad$ $x = \sqrt{9}$

$x = \sqrt{9}i$ $\qquad$ $x = \pm 3$

$x = \pm 3i$

Hence, four fourth roots of 81 are: $3, -3, 3i, -3i$

(iii) 625

Solution:

Let x be a fourth root of 625, therefore,

$x = \sqrt[4]{625}$

$x = 625^{\frac{1}{4}}$

$(x)^4 = \left(625^{\frac{1}{4}}\right)^4$

$x^4 - 625 = 0$

$\left(x^2\right)^2 - (25)^2 = 0$

$(x^2+25)(x^2-25) = 0$ $\qquad$ Q $a^2 - b^2 = (a+b)(a-b)$

$x^2 + 25 = 0$ $\qquad$ $x^2 - 25 = 0$

$x^2 = -25$ $\qquad$ $x^2 = 25$

$x = \sqrt{-25}$ $\qquad$ $x = \sqrt{25}$

$x = \sqrt{25}i$ $\qquad$ $x = \pm 5$

$x = \pm 5i$

Hence, four fourth roots of 625 are: $5, -5, 5i, -5i$

8. Solve the following equations:

(i) $2x^4 - 32 = 0$

Solution:

$2x^4 - 32 = 0$

$2(x^4 - 16) = 0$

$x^4 - 16 = 0$

$(x^2)^2 - (4)^2 = 0$

$(x^2 + 4)(x^2 - 4) = 0$ $\qquad$ Q $a^2 - b^2 = (a+b)(a-b)$

$x^2 + 4 = 0$ $\qquad$ $x^2 - 4 = 0$

$x^2 = -4$ $\qquad$ $x^2 = 4$

$x = \sqrt{-4}$ $\qquad$ $x = \sqrt{4}$

$x = \sqrt{4}i$ $\qquad$ $x = \pm 2$

$x = \pm 2i$

Hence, solution set = $\{2, -2, 2i, -2i\}$

(ii) $3y^5 - 243y = 0$

Solution:

$3y^5 - 243y = 0$

$3y(y^4 - 81) = 0$

$3y = 0$ $\qquad$ $y^4 - 81 = 0$

$y = 0$ $\qquad$ $(y^2)^2 - (9)^2 = 0$

$(y^2 + 9)(y^2 - 9) = 0$ $\qquad$ Q $a^2 - b^2 = (a+b)(a-b)$

$y^2 + 9 = 0$ $\qquad$ $y^2 - 9 = 0$

$y^2 = -9$ $\qquad$ $y^2 = 9$

$y = \pm\sqrt{-9}$ $\qquad$ $y = \pm\sqrt{9}$

$y = \pm 3i$ $\qquad$ $y = \pm 3$

Hence, solution set = $\{0, 3, -3, 3i, -3i\}$

(iii) $x^3 + x^2 + x + 1 = 0$

Solution:

$x^3 + x^2 + x + 1 = 0$

$x^2(x+1) + 1(x+1) = 0$

$(x+1)(x^2+1) = 0$

$x + 1 = 0$ $\qquad$ $x^2 + 1 = 0$

$x = -1$ $\qquad$ $x^2 = -1$

$x = \pm\sqrt{-1}$ $\qquad$ Q $i^2 = -1$

$x = \pm\sqrt{i^2}$

$x = \pm i$

Hence, solution set = $\{-1, i, -i\}$

(iv) $5x^5 - 5x = 0$

Solution:

$5x^5 - 5x = 0$

$5x(x^4 - 1) = 0$

$5x = 0$ $\quad$ $x^4 - 1 = 0$

$x = 0$ $\quad$ $(x^2)^2 - (1)^2 = 0$

$(x^2 + 1)(x^2 - 1) = 0$

$x^2 + 1 = 0$ $\quad$ $x^2 - 1 = 0$

$x^2 = -1$ $\quad$ $x^2 = 1$

$x = \pm i$ $\quad$ $x = \pm 1$

Hence, solution set = $\{0, 1, -1, i, -i\}$